M70 12.1 - 2nd & 3rd days

Lesson 33
Lesson 34

12.1 - 1st Composition
12.1 - 2nd Inverse Functions
12.1 - 3rd Inverse Functions

Math 70 12.1 Inverse Functions

<u>Objectives</u>

1) Observe that inverse functions, when composed, "un-do" each other.
    a) Notation $f^{-1}(x)$ is pronounce "f-inverse of x".

CAUTION: This notation looks like an exponent, but it's not. Exponent would be outside: $[f(x)]^{-1} = \dfrac{1}{f(x)}$

CAUTION: $f^{-1}(x)$ is NOT usually the reciprocal of $f(x)$.

    b) "Un-do" means: $(f^{-1} \circ f)(x) = x$, $(f \circ f^{-1})(x) = x$
2) Find the inverse of a function.
    a) From a list of ordered pairs
    b) Algebraically
    c) Is the resulting inverse a function?
3) Determine if a function has an inverse function, AKA "is an invertible function". A function has an inverse function if:
    a) it is one-to-one: for each y value there is at most one x value.
    b) its graph passes the horizontal line test.
    c) Recall: To be a function, the graph must pass the vertical line test.

CAUTION: A graph can be one-to-one and not be a function, or vice-versa. To be an "invertible function", it must pass both the VLT and the HLT.

4) Graph functions and their inverses.
5) Use algebra to show that two functions are inverses of each other. Show that: $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$

Practice and Examples

1) Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x-3)$, find:
    a) $(g \circ f)(x)$

    b) $(f \circ g)(x)$

    c) $(g \circ f)(23)$

2) Complete the tables for $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$

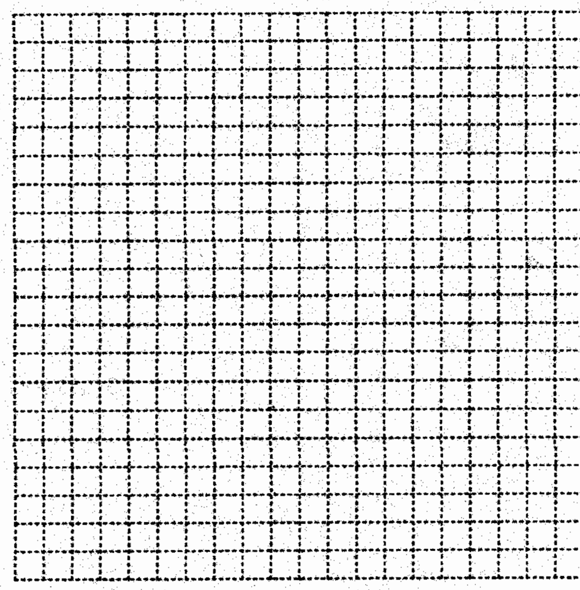
x	y=f(x)
-2	
0	
1	
52	
-1	
3	
5	
107	

x	y= g(x)
-1	
3	
5	
107	
-2	
0	
1	
52	

3) Find the inverse of the function. Graph the points of the function. Is the inverse a function?

x	y=f(x)
0	2
1	3
2	4
3	5

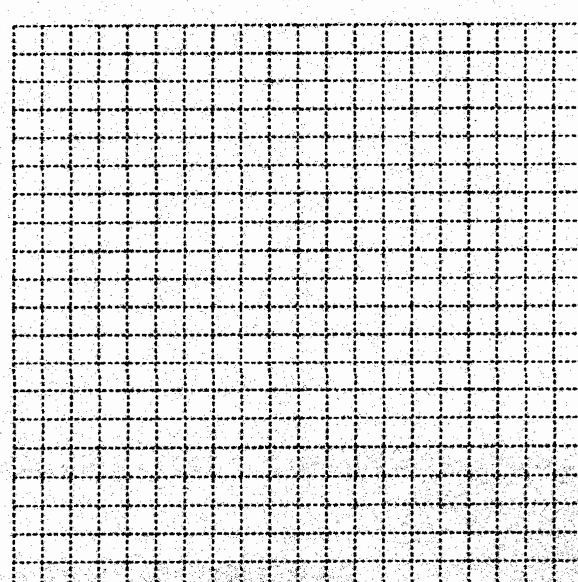
x	y



4) Find the inverse of the function. Graph the points of the function. Is the inverse a function?

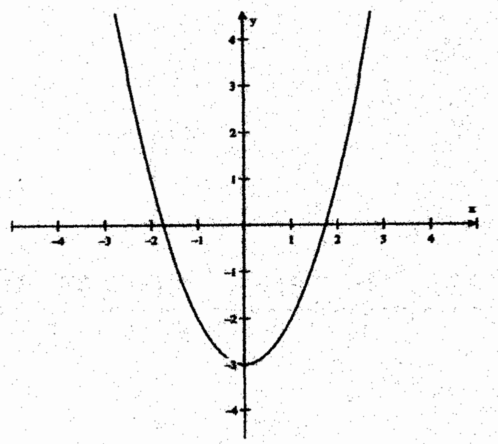
x	y=f(x)
-1	2
1	3
2	4
5	3

x	y



For each graph, identify

a) Is it a function?
b) Is it one-to-one?
c) Is it an invertible function?



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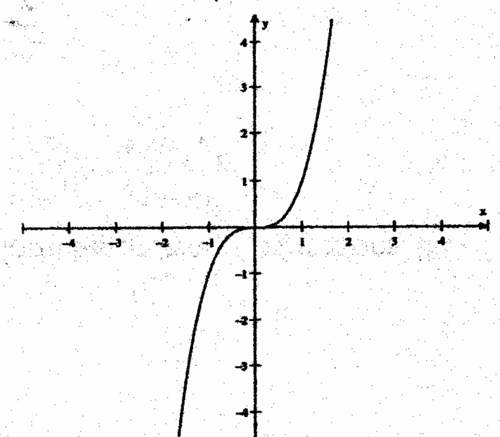

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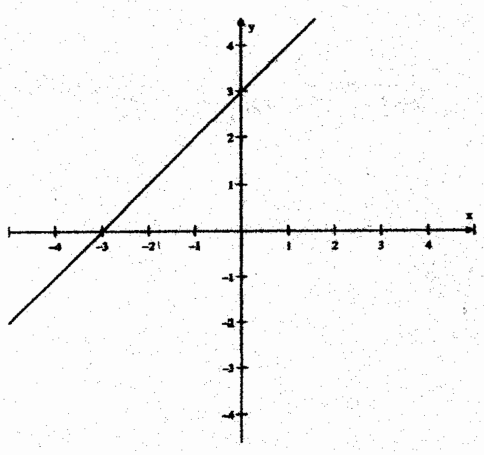

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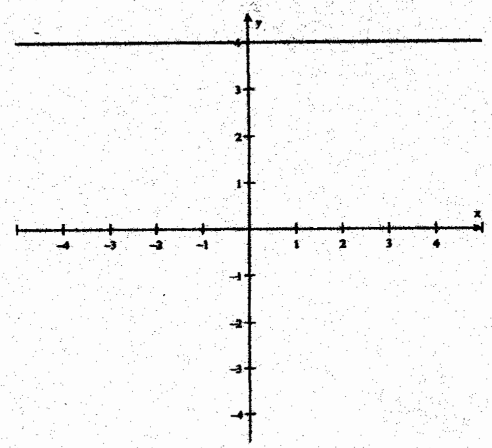

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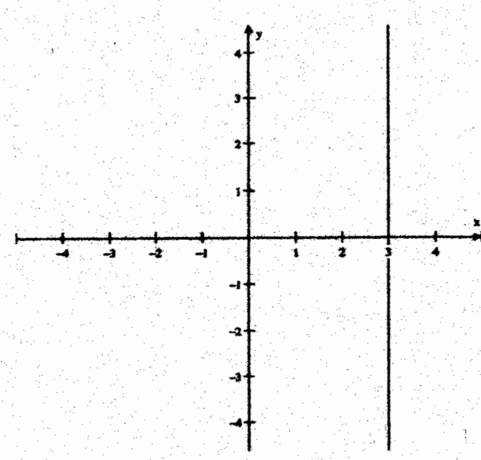

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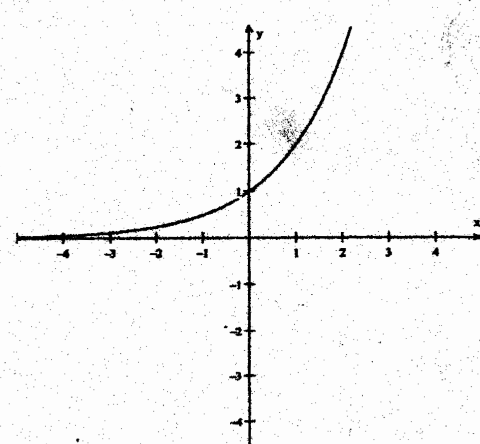


10)

11) Find the inverse of the function.

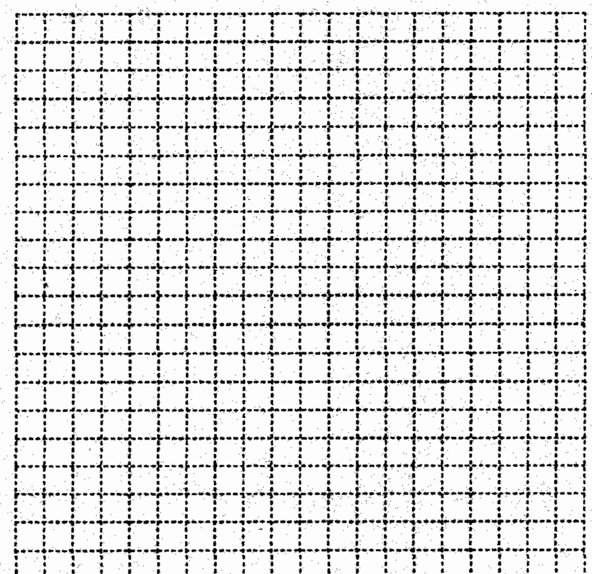
a) $f(x)=2x+3$

b) $f(x)=4x^3-1$

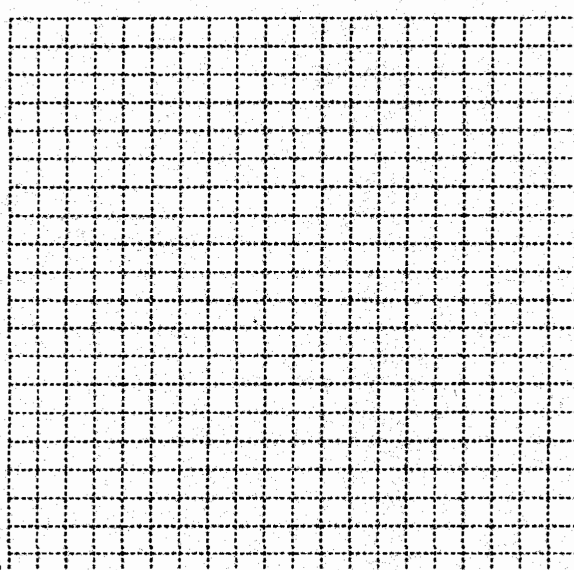
c) $f(x)=\sqrt[3]{5x+7}$

d) $f(x)=\dfrac{8}{2x-3}$

12) Graph the function and its inverse on the same grid.



a) $f(x)=2x-5$



b) $f(x)=(x+5)^3$

The result we just got is strange and unusual!

1) $(f \circ g)(x)$ and $(g \circ f)(x)$ are usually different; but with this $f(x)$ and this $g(x)$, we get the same result, x.

2) $(f \circ g)(x)$ and $(g \circ f)(x)$ are usually more complicated expressions, yet this time we get a very simple result, x.

3) When we put in a value of x, like $x = 23$, we see that $f(23)$ does something to change 23 to a new result, 49,

But... $g(49)$ un-does that, to go back to 23.

Two functions that have this relationship

$$f(g(x)) = x$$
$$g(f(x)) = x$$

are called inverses of each other, or inverse functions.

In particular,

$f(x)$ is the inverse of $g(x)$
$g(x)$ is the inverse of $f(x)$

Notation for this special inverse relationship:

$f(x) = g^{-1}(x)$ "g-inverse-of x"
$g(x) = f^{-1}(x)$ "f-inverse-of x"

<u>CAUTION!</u> This is not an exponent! Not $[f(x)]^{-1} = \frac{1}{f(x)}$.

2) Complete the tables for $f(x) = 2x+3$ and $g(x) = \frac{1}{2}(x-3)$

x	y=f(x)
-2	-1
0	3
1	5
52	107
-1	1
3	9
5	13
107	217

(-2, -1)

other ordered pairs on f(x):
(-2.5, -2)
(-1.5, 0)
(24.5, 52)

$f^{-1}(x) = g(x)$

x	y=~~f(x)~~
-1	-2
3	0
5	1
107	52
-2	-2.5
0	-1.5
1	-1
52	24.5

(-1, -2)

These ordered pairs just swap x ⟺ y.

other ordered pairs on $f^{-1}(x)$
(9, 3)
(13, 5)
(217, 107)

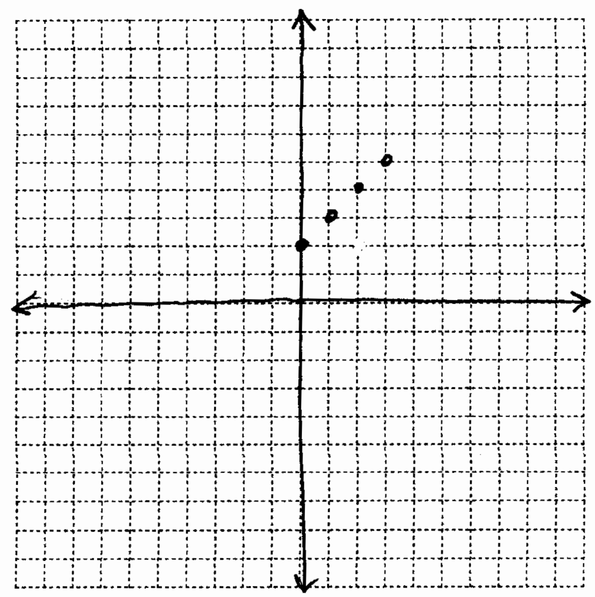
3) Find the inverse of the function. Graph the points of the function. Is the inverse a function?

x	y=f(x)
0	2
1	3
2	4
3	5

swap the locations of x and y.

x	y
2	0
3	1
4	2
5	3

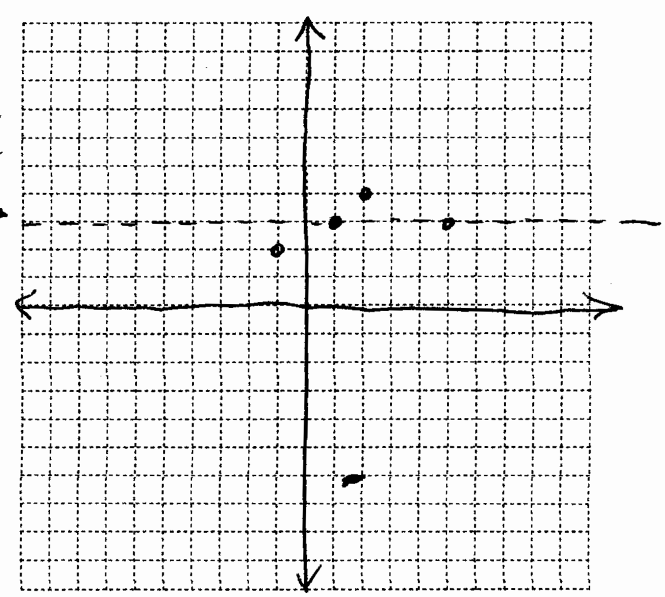
Yes, the inverse is a function.



4) Find the inverse of the function. Graph the points of the function. Is the inverse a function?

x	y=f(x)
-1	2
1	3
2	4
5	3

If we can draw any horizontal line that crosses the graph of f more than once, the inverse will not be a function.



x	y
2	-1
3	1
4	2
3	5

no, the inverse is not a function because x=3 has two y-values, 1 and 5

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Horizontal Line Test (H.L.T.)

A function passes the horizontal line test if:
every possible imaginary horizontal line crosses the graph at at most one point.

If a graph passes the horizontal line test, we say that it is one-to-one.

If a graph passes the vertical line test, we say that it is a function.

If a graph passes both the H.L.T. and the V.L.T., We say that it is an invertible function.

f passes VLT $\Rightarrow$ f is a function $\rightarrow$

f passes H.L.T $\Rightarrow$ f is one-to-one $\Rightarrow$ f's inverse is a function $\rightarrow$

if f passes both V.L.T. and H.L.T, it is an invertible function.

While it is always possible to swap the x and y Coordinates to get an inverse,
if the result we get by doing so is not a function, we say it is not an invertible function.

For each graph, identify

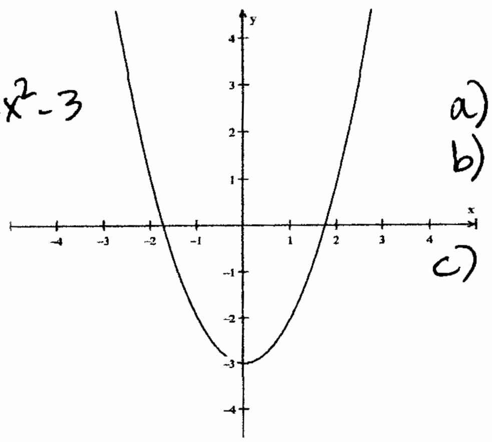
a) Is it a function? = Does it pass the VLT?

b) Is it one-to-one? = Does it pass the HLT?

c) Is it an invertible function? = Does it pass both the VLT <u>and</u> the HLT?

5)

$y = x^2 - 3$



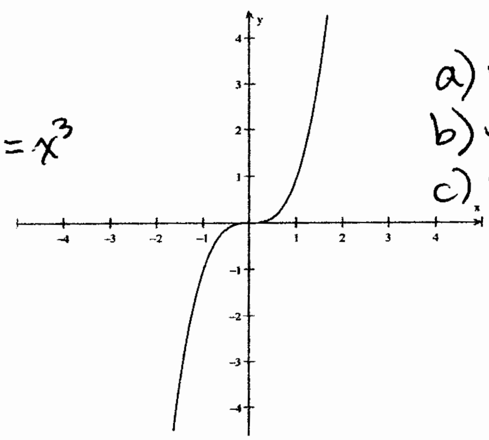
a) yes VLT

b) no HLT

c) no

6)

$y = x^3$



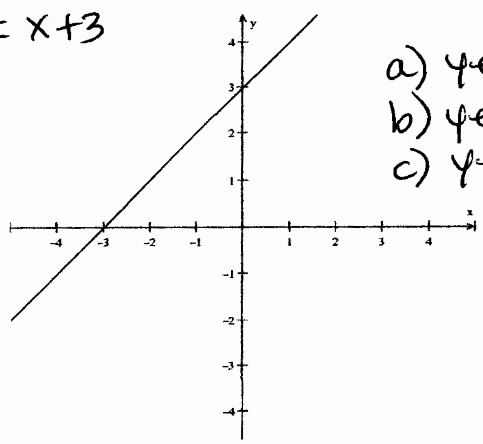
a) yes -VLT

b) yes HLT

c) yes.

7)

$y = x + 3$



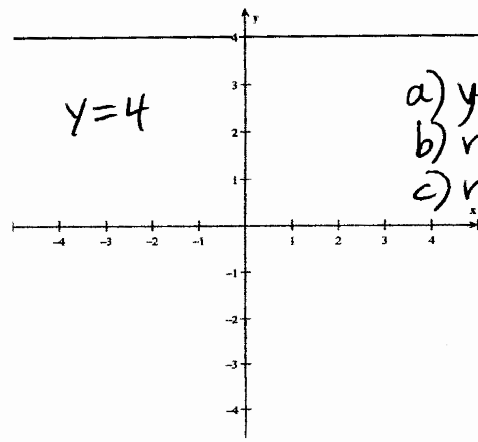
a) yes

b) yes

c) yes

8)

$y = 4$



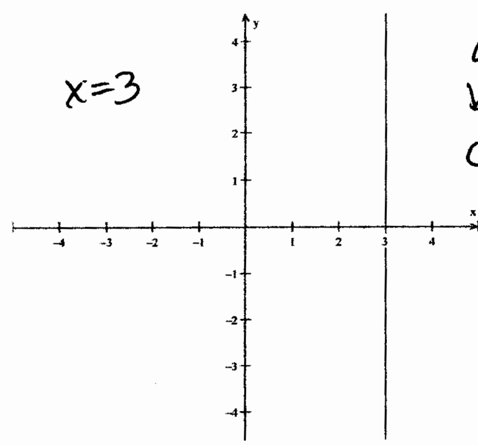
a) yes

b) no HLT

c) no

9)

$x = 3$



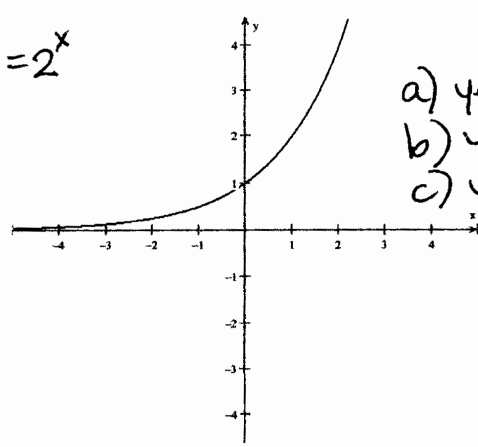
a) no-VLT

b) yes

c) no.

10)

$y = 2^x$



a) yes

b) yes

c) yes

11) Find the inverse of the function.

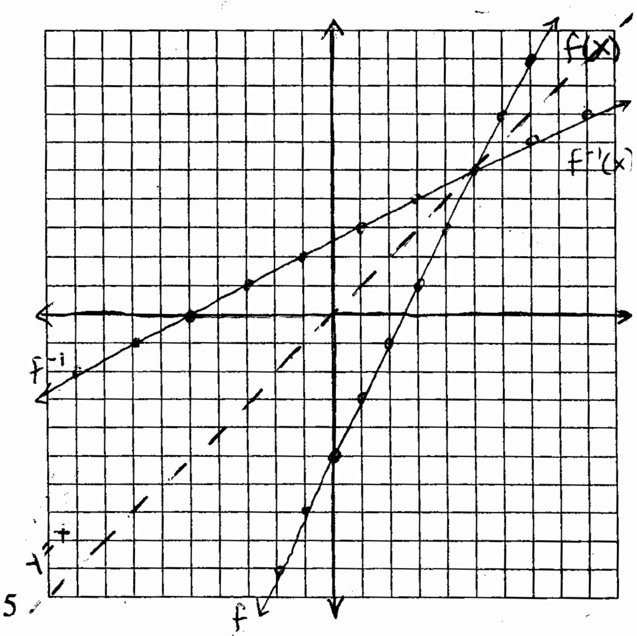
a) $f(x)=2x+3$ see next page

b) $f(x)=4x^3-1$

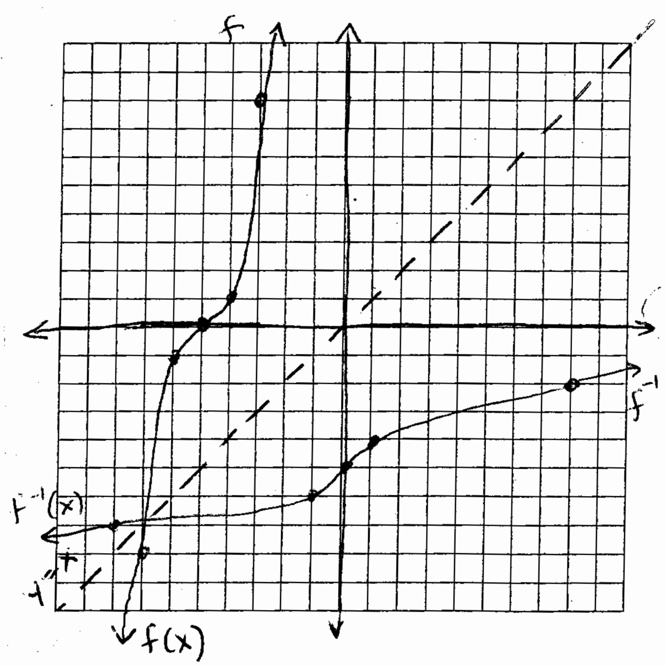
c) $f(x)=\sqrt[3]{5x+7}$

d) $f(x)=\dfrac{8}{2x-3}$

12) Graph the function and its inverse on the same grid.



a) $f(x)=2x-5$



b) $f(x)=(x+5)^3$

Notice that swapping the x and y coordinates creates a graph that is a reflection of the original graph.

This reflection is always a mirror image across the diagonal line y=x.

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11 a) $f(x) = 2x+3$

step 1 $y = 2x+3$

step 2 $x = 2y+3$

step 3 $x-3 = 2y$

$\frac{x-3}{2} = y$

$y = \frac{x-3}{2}$

step 4 $\boxed{f^{-1}(x) = \frac{x-3}{2}}$

To find inverse:

Step 1: Replace $f(x)$ by y.

step 2: Swap x & y

step 3: Isolate y

step 4: Replace y by $f^{-1}(x)$.

b) $f(x) = 4x^3 - 1$

$y = 4x^3 - 1$ step 1

$x = 4y^3 - 1$ step 2

$\frac{x+1}{4} = \frac{4y^3}{4}$ step 3

$\frac{x+1}{4} = y^3$

$\sqrt[3]{\frac{x+1}{4}} = y$

To remove exp 3, cube root both sides.

$\frac{\sqrt[3]{x+1}}{\sqrt[3]{4}} = y$

$y = \frac{\sqrt[3]{x+1}}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}}$ ← To rationalize a cube root, need a perfect cube, $\sqrt[3]{2^3} = \sqrt[3]{8} = \sqrt[3]{4} \cdot \sqrt[3]{2}$

$\boxed{f^{-1}(x) = \frac{\sqrt[3]{2(x+1)}}{2}}$ step 4

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c) $f(x) = \sqrt[3]{5x+7}$

step 1 $y = \sqrt[3]{5x+7}$

step 2 $x = \sqrt[3]{5y+7}$ ← To remove a $\sqrt[3]{}$, cube both sides.

step 3 $x^3 = 5y+7$

$$\frac{x^3-7}{5} = \frac{5y}{5}$$

$y = \frac{1}{5}(x^3-7)$ or $y = \frac{x^3-7}{5}$

$f^{-1}(x) = \frac{1}{5}(x^3-7)$ $\boxed{f^{-1}(x) = \frac{x^3-7}{5}}$

step 4 or $\boxed{f^{-1}(x) = \frac{1}{5}x^3 - \frac{7}{5}}$

d) $f(x) = \frac{8}{2x-3}$

step 1 $y = \frac{8}{2x-3}$

step 2 $x = \frac{8}{2y-3}$ cross-multiply to clear fractions

$x(2y-3) = 8$ divide by x to both sides

$2y-3 = \frac{8}{x}$

$2y = \frac{8}{x}+3$

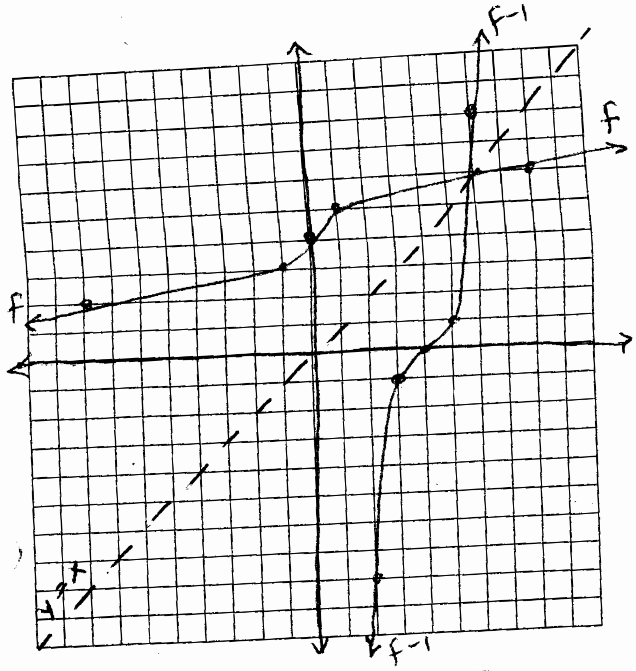
$y = \frac{1}{2}\left(\frac{8}{x}+3\right)$ distribute

$\boxed{f^{-1}(x) = \frac{4}{x} + \frac{3}{2}}$

Math 70 Additional Examples

① $f(x) = \sqrt[3]{x} + 4$

$f^{-1}(x) = (x-4)^3$



② $f(x) = \dfrac{3}{x-2}$

$y = \dfrac{3}{x-2}$

$x = \dfrac{3}{y-2}$

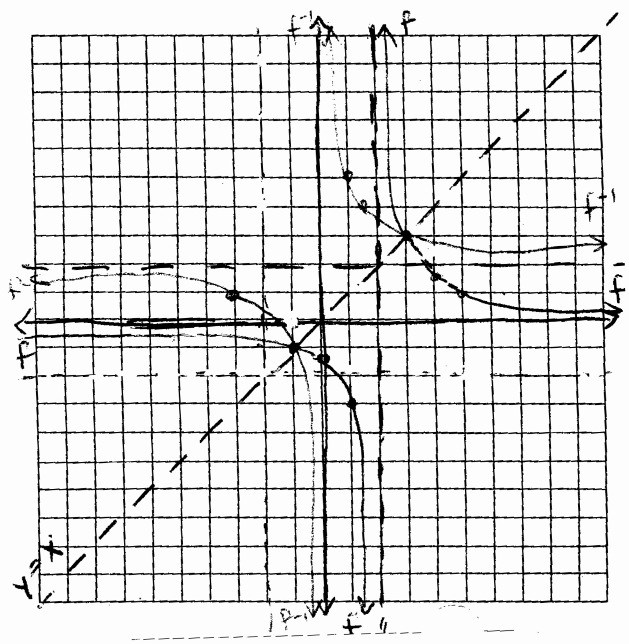
$x(y-2) = 3$

$xy - 2x = 3$

$xy = 2x + 3$

$y = \dfrac{2x+3}{x}$

$f^{-1}(x) = \dfrac{2x+3}{x}$



Math 70 9.2 Inverse Functions

* Practice this even if MathXL can't make you do it! *

3) Determine if $f(x) = x^3+2$ and $g(x) = \sqrt[3]{x-2}$ are inverses of each other.

Functions which are inverses "un-do" each other → regardless of which function is first.

If $f(x) = x^3+2$ then $f^{-1}(x) = \sqrt[3]{x-2}$.
If $f(x) = \sqrt[3]{x-2}$ then $f^{-1}(x) = x^3+2$.

To demonstrate that two functions are inverses, must show two things

1) $f(g(x)) = x$
2) $g(f(x)) = x$

x	$f(x)$
3	29
1	3

x	$f^{-1}(x)$
29	3
3	1

$$f(g(x)) = \left(\sqrt[3]{x-2}\right)^3 + 2$$
$$= x - 2 + 2$$
$$= x \quad ✓$$

IMPORTANT: The work is the answer to this type of question

$$g(f(x)) = \sqrt[3]{(x^3+2)-2}$$
$$= \sqrt[3]{x^3}$$
$$= x \quad ✓$$

$f(g(x)) = x$ ← may make more sense if we demonstrate with a value for x.

If $x=1$

$g(1) = \sqrt[3]{1-2} = \sqrt[3]{-1} = -1$

$f(-1) = (-1)^3 + 2 = -1+2 = 1$ ← same value, $x=1$ as at start. $f(g(1)) = 1$

Similarly for $g(f(x)) = x$:

If $x=1$.

$f(1) = 1^3+2 = 3$

$g(3) = \sqrt[3]{3-2} = \sqrt[3]{1} = 1$ ← same value $x=1$ as at start. $g(f(1)) = 1$

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9.1.27 If $f(x) = x^2 + 8$, $g(x) = \sqrt{x}$ and $h(x) = 2x$, write $F(x) = 4x^2 + 8$ as a composition using two of the given functions.

$F(x) = (\square \circ \square)(x)$

Given $f(x) = x^2+8$
$g(x) = \sqrt{x}$
$h(x) = 2x$.

Rewrite $F(x) = 4x^2+8$ as a composition of two functions.

Notice: $g(x) = \sqrt{x}$ has a square root, but
$F(x) = 4x^2+8$ has no square root.

So we probably aren't going to use $g(x)$ at all.

We'll use $f(x) = x^2+8$
$h(x) = 2x$.

It's either $(f \circ h)(x)$ or $(h \circ f)(x)$.

Work out what these are:

$(f \circ h)(x)$	$(h \circ f)(x)$
$= f(h(x))$	$= h(f(x))$
$= (2x)^2 + 8$	$= 2(x^2+8)$
$= 4x^2+8$	$= 2x^2+16$

↑

This is what we wanted.

$$\boxed{F(x) = (f \circ h)(x).}$$